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Note

Sharpness of Timan's Converse Result for Polynomial Approximation*

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We verify a conjecture of M. Hasson about the sharpness of Timan's converse result. © 1991 Academic Press, Inc.

A. F. Timan [2, 6.2.3] proved that if for a function f that is continuous on $[-1, 1]$ there are polynomials P_n, $\deg P_n \leqslant n$, $n = 1, 2, \ldots$, with the property

$$|f(x) - P_n(x)| \leqslant \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^r \omega\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right), \qquad -1 \leqslant x \leqslant 1,$$

where the modulus of continuity ω satisfies

$$\sum_{n=1}^{\infty} \frac{1}{n}\,\omega\left(\frac{1}{n}\right) < \infty,$$

then f is r-times continuously differentiable on $[-1, 1]$. In [1] M. Hasson stated the following conjecture: Suppose that $\{a_l\}$ is a monotone increasing positive sequence with

$$\sum_{l=1}^{\infty} \frac{1}{l a_l} = \infty.$$

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Then there exists a continuous function f which is not in $C^{(r)}[-1, 1]$, but for its best polynomial approximation $E_n(f)$ by polynomials of degree at most n we have

$$E_n(f) = O\left(\frac{1}{n^{2r}a_n}\right). \tag{1}$$

This would show that the Timan result cannot be improved in any sense. Hasson himself verified the conjecture for $r=1$, but the general case has remainded open.

In this note we give a simple proof for the above conjecture. We show that for some sequence $\{\varepsilon_l\}$ of the numbers $\{0, 1\}$ the function

$$f(x) = \sum_{l=1}^{\infty} \varepsilon_l \frac{1}{l^{2r+1}a_l} T_l(x),$$

where $T_l(x) = \cos(l \arccos x)$ are the Chebyshev polynomials, satisfies the above conditions. Since

$$\sum_{l=n+1}^{\infty} \frac{1}{l^{2r+1}a_l} \leqslant \frac{1}{n^{2r}a_n},$$

(1) is obvious for any choice of the ε_k's. The heuristics behind $f \notin C^r[-1, 1]$ is that by

$$T_n^{(r)}(1) = \frac{n^2(n^2-1)\cdots(n^2-(r-1)^2)}{1\cdot 3\cdots(2r-1)} > c_r n^{2r}, \qquad n \geqslant r$$

(see [2, 4.8.8]) we get by formally differentiating the series representing f termwise that if all ε_k equals 1, then

$$f^{(r)}(1) \geqslant \sum_{l=1}^{\infty} \frac{1}{l^{2r+1}a_l} c_r l^2 = \infty.$$

Of course, termwise differentiation is not permitted; this is why we need the following argument.

We shall choose sequences $\{M_k\}$ and $\{N_k\}$ of natural numbers so that $M_k < N_k < M_{k+1}$ is satisfied for all k and set $\varepsilon_l = 1$ for $M_k \leqslant l < N_k$, $k = 1, 2, \ldots$, and $\varepsilon_l = 0$ otherwise. We can differentiate termwise the series representing f $(r-1)$-times. Hence, we can select M_k, N_k, and small positive numbers τ_k in the order ..., M_k, N_k, τ_k, M_{k+1}, ... in such a way that with

$$f_k(x) = \sum_{l=1}^{M_k} \varepsilon_l \frac{1}{l^{2r+1}a_l} T_l(x),$$

we satisfy the following conditions one after the other:

$$f_k^{(r)}(1)=\sum_{l=1}^{M_k}\frac{1}{l^{2r+1}a_l}\frac{l^2(l^2-1)\cdots(l^2-(r-1)^2)}{1\cdot 3\cdots(2r-1)}>k+1,$$

$$\frac{f_k^{(r-1)}(1)-f_k^{(r-1)}(1-\tau_k)}{\tau_k}>k+1,$$

$$\sum_{l=N_k}^{\infty}\frac{1}{l^3a_l}<\frac{\tau_k}{2}.$$

With this construction we obviously have

$$\frac{f^{(r-1)}(1)-f^{(r-1)}(1-\tau_k)}{\tau_k}\geqslant\frac{f_k^{(r-1)}(1)-f_k^{(r-1)}(1-\tau_k)}{\tau_k}-1>k.$$

That is, $f^{(r-1)}$ is not differentiable at 1 from the left.

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References

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2. A. F. Timan, "Theory of Approximation of Functions of a Real Variable," Macmillan Co., New York, 1963.